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Abstract

In this paper, we study the (geodesic) hull number of graphs. For any two vertices $u, v \in V$ of a connected undirected graph $G = (V, E)$, the closed interval $I[u, v]$ of u and v is the set of vertices that belong to some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is (geodesically) convex if $I[S] = S$. Given a subset $S \subseteq V$, the convex hull $I_h[S]$ of S is the smallest convex set that contains S. We say that S is a hull set of G if $I_h[S] = V$. The size of a minimum hull set of G is the hull number of G, denoted by $hn(G)$.

First, we show a polynomial-time algorithm to compute the hull number of any P_5-free triangle-free graph. Then, we present four reduction rules based on vertices with the same neighborhood. We use these reduction rules to propose a fixed parameter tractable algorithm to compute the hull number of any graph G, where the parameter can be the size of a vertex cover of G or, more generally, its neighborhood diversity, and we also use these reductions to characterize the hull number of the lexicographic product of any two graphs.

Keywords: Graph Convexity, Hull Number, Geodesic Convexity, P_5-free Graphs, Lexicographic Product, Parameterized Complexity, Neighborhood Diversity.

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1 Introduction

All graphs in this work are undirected, simple and loop-less. Given a connected graph $G = (V, E)$, the closed interval $I[u, v]$ of any two vertices $u, v \in V$ is the set of vertices that belong to some u-v geodesic of G, i.e. some shortest (u, v)-path. For any $S \subseteq V$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. A subset $S \subseteq V$ is *(geodesically) convex* if $I[S] = S$. Given a subset $S \subseteq V$, the *convex hull* $I_h[S]$ of S is the smallest convex set that contains S. We say that a vertex v is *generated* by a set of vertices S if $v \in I_h[S]$. Equivalenty, given a set S, let $I^0[S] = S$ and $I^k[S] = I[I^{k-1}[S]]$, for $k > 0$. We say that v is generated by S *at step* $t \geq 1$, if $v \in I^t[S]$ and $v \notin I^{t-1}[S]$. Observe that the convex hull $I_h(S)$ of S is equal to $I^{|V(G)|}[S]$. We say that S is a *hull set* of G if $I_h[S] = V$. The size of a minimum hull set of G is the *hull number* of G, denoted by $hn(G)$ [9].

It is known that computing $hn(G)$ is an NP-hard problem for bipartite graphs [3]. Several bounds on the hull number of triangle-free graphs are presented in [8]. In [7], the authors show, among other results, that the hull number of any P_4-free graph, i.e. any graph without induced path with four vertices, can be computed in polynomial time. In Section 3, we show a linear-time algorithm to compute the hull number of any P_5-free triangle-free graph.

In Section 4, we show four reduction rules to obtain, from a graph G, another graph G^* that has one vertex less than G and which satisfies either $hn(G) = hn(G^*)$ or $hn(G) = hn(G^*) + 1$, according to the used rule. We then first use these rules to obtain a fixed parameter tractable (FPT) algorithm, where the parameter is the neighborhood diversity of the input graph. For definitions on Parameterized Complexity we refer to [10]. Given a graph G and vertices $u, v \in V(G)$, we say that u and v are *twins* (a.k.a. of the *same type*) if $N(v) \setminus \{u\} = N(u) \setminus \{v\}$. The *neighborhood diversity* of a graph is k, if its vertex set can be partitioned into k sets $S_1, \ldots, S_k$, such that any pair of vertices $u, v \in S_i$ are twins. This parameter was proposed by Lampis [12], motivated by the fact that a graph of bounded vertex cover also has bounded neighborhood diversity, and therefore the later parameter can be used to obtain more general results. Many problems have been shown to be FPT when the parameter is the neighborhood diversity [11].

Finally, we use these rules to characterize the hull number of the lexicographic product of any two graphs. Given two graphs G and H, the *lexicographic product* $G \circ H$ is the graph whose vertex set is $V(G \circ H) = V(G) \times V(H)$ and such that two vertices (g_1, h_1) and (g_2, h_2) are adjacent if, and only if, either $g_1g_2 \in E(G)$ or we have that both $g_1 = g_2$ and $h_1h_2 \in E(G)$.

It is known in the literature a characterization of the (geodesic) convex sets in the lexicographic product of two graphs [1] and a study of the pre-hull number for this product [13]. There are also some results concerning the hull number of the Cartesian and strong products of graphs [5, 6].

2 Preliminaries

Let us recall some definitions and lemmas that we use in the sequel.

We denote by $N_G(v)$ (or simply $N(v)$) the neighborhood of a vertex. A vertex v is *simplicial* (resp. *universal*) if $N(v)$ is a clique (resp. is equal to $V(G) \setminus \{v\}$). Let $d_G(u,v)$ denote the *distance* between u and v, i.e. the length of a shortest (u,v)-path. A subgraph $H \subseteq G$ is *isometric* if, for each $u,v \in V(H)$, $d_H(u,v) = d_G(u,v)$. A P_k (resp. C_k) in a graph G denotes an induced path (resp. cycle) on k vertices. Given a graph H, we say that a graph G is H-free if G does not contain H as an induced subgraph. Moreover, we consider that all the graphs in this work are connected. Indeed, if a graph G is not connected, its hull number can be computed by the sum of the hull numbers of its connected components, as observed by Dourado et al. [7].

Lemma 1. *[9] For any hull set S of a graph G, S contains all simplicial vertices of G.*

Lemma 2. *[7] Let G be a graph which is not complete. No hull set of G with cardinality $hn(G)$ contains a universal vertex.*

Lemma 3. *[7] Let G be a graph, H be an isometric subgraph of G and S be any hull set of H. Then, the convex hull of S in G contains $V(H)$.*

Lemma 4. *[7] Let G be a graph and S a proper and non-empty subset of $V(G)$. If $V(G) \setminus S$ is convex, then every hull set of G contains at least one vertex of S.*

3 Hull number of P_5-free triangle-free graphs

In this section, we present a linear-time algorithm to compute $hn(G)$, for any P_5-free triangle-free graph G. To prove the correctness of this algorithm, we need to recall some definitions and previous results:

Definition 1. *Given a graph $G = (V,E)$, we say that $S \subseteq V$ is a* dominating set *if every vertex $v \in V \setminus S$ has a neighbor in S.*

It is well known that:

Theorem 1. *[4] G is P_5-free if, and only if, for every induced subgraph $H \subseteq G$ either $V(H)$ contains a dominating C_5 or a dominating clique.*

As a consequence, we have that:

Corollary 1. *If G is a connected P_5-free bipartite graph, then there exists a dominating edge in G.*

Theorem 2. *The hull number of a P_5-free bipartite graph $G = (A \cup B, E)$ can be computed in linear time.*

For the next result, recall that the complexity of finding the convex hull of a set of vertices $S \subseteq V(G)$ of a graph G is $\mathcal{O}(|S||E(G)|)$, as described in [7]. We can relax the constraint of G being bipartite to obtain the following:

Corollary 2. *If G is a P_5-free triangle-free graph, then $hn(G)$ can be computed in polynomial time.*

4 Neighborhood Diversity and Lexicographic Product

In this section, we present four reduction rules to compute the hull number of a graph. We need to introduce some definitions.

Given a graph G, we say that two vertices v and v' are *twins* if $N(v)\setminus\{v'\} = N(v') \setminus \{v\}$. If v and v' are adjacent, we call them *true* twins, otherwise we say that they are *false* twins.

Let G be a graph and v and v' be two of its vertices. The *identification* of v' into v is the operation that produces a graph G' such that $V(G') = V(G)\setminus\{v'\}$ and $E(G') = (E(G) \setminus \{v'w \mid w \in N_G(v')\}) \cup \{vw \mid v'w \in E(G) \text{ and } w \neq v\}$.

Lemma 5. *Let G be a graph and v and v' be non-simplicial and twin vertices. Let G' be obtained from G by the identification of v' into v. Then, $hn(G) = hn(G')$.*

Lemma 6. *Let G be a graph and v, v', v'' be simplicial and pairwise false twin vertices. Let G' be obtained from G by the identification of v'' into v. Then, $hn(G) = hn(G') + 1$.*

Observe that we cannot simplify the statement of Lemma 6 to consider any pair of simplicial false twin vertices instead of triples. As an example, consider the graph obtained by removing an edge uv from a complete graph with more than 3 vertices.

Lemma 7. *Let G be a graph and v, v' be simplicial and true twin vertices. Let G' be obtained from G by the identification of v' into v. Then, $hn(G) = hn(G') + 1$.*

Recall that the *neighborhood diversity* of a graph is k, if its vertex set can be partitioned into k sets $S_1, \ldots, S_k$, such that any pair of vertices $u, v \in S_i$ are twins. Now, we use this partition to obtain the following result:

Theorem 3. *Let G be a graph whose neighborhood diversity is at most k. Then, there exists an FPT algorithm to compute $hn(G)$ in $\mathcal{O}(4^k poly(|V(G)|))$-time.*

As pointed before, a graph of bounded vertex cover size has also bounded neighborhood diversity, therefore the previous result also holds for this parameter.

Now, we use Lemma 5 and Lemma 7 to determine the lexicographic product of two graphs. Let $S(G)$ denote the set of simplicial vertices of G.

Observe that if G has a single vertex, then $hn(G \circ H) = hn(H)$. Else, we have that:

Theorem 4. *Let G be a connected graph, such that $|V(G)| \geq 2$, and let H be an arbitrary graph. Then,*

$$hn(G \circ H) = \begin{cases} 2, & \textit{if } H \textit{ is not complete;} \\ (|V(H)|-1)|S(G)| + hn(G), & \textit{otherwise.} \end{cases}$$

5 Conclusions

In this work, we first presented a linear-time algorithm to compute the hull number of any P_5-free triangle-free graph. However, the computational complexity of determining the hull number of a P_5-free graph is still unknown. More generally, we propose the following open question:

Question 1. *For a fixed k, what is the computational complexity of determining $hn(G)$, for a P_k-free graph G?*

In the second part of this paper, we introduced four reduction rules that we use to present an FPT algorithm to compute the hull number of any graph, where the parameter is its neighborhood diversity, and a characterization of the lexicographic product of any two graphs. It is already known in the literature another FPT algorithm to compute the hull number of any graph, where the parameter is the number of its induced P_4's [2]. To the best our knowledge, the following is also open:

Question 2. *Given a graph G, is there an FPT algorithm to determine whether $hn(G) \leq k$, for a fixed k?*

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